$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \frac{1}{9}\int_0^{\infty} x^2 \cdot e^{-x/3} \cdot dx$$

$$= \frac{1}{9}\left[x^2\left(\frac{e^{-x/3}}{-1/3}\right) - (2x)\left(\frac{e^{-x/3}}{1/9}\right) + 2\left(\frac{e^{-x/3}}{-1/27}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}[0 - 0 + 0 + 54] = 6$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\int_6^{\infty} x \cdot e^{-x/3} \cdot dx = \frac{1}{9}\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_6^{\infty}$$

$$= \frac{1}{9}\left[(0 - 0) - (-18\, e^{-2} - 9e^{-2})\right] = 3e^{-2} = 0.406$$

Exercise - I

1. From an urn containing 3 red balls and 2 white balls, a man is to draw 2 balls at random replacement. He gets ₹ 20 for each red ball and ₹ 10 for each white ball he draws. Find his expec
[Ans :
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3. Expectation of a

We can now extend th variable.

1. Definition : Let X $p_2, \ldots\, p_n$ and $g(X)$ be a fu

2. Definition : Let that $g(X)$ is a random va

Note ....

1. If $g(X) = aX$

And E[

e.g. E

e.g. E

2. If $g(X) =$

3. It should b

4. Putting

$$= a^2 E[X - E(X$$

$$\boxed{V(aX) = a^2 V(X)}$$

Cor. 1 :

Putting $b = 0$, we get the result.

$$\boxed{V(X + b) = V(X)}$$

Cor. 2 :

Putting $a = 1$, we get the result.

Note ....

Note that although $E(aX + b) = aE(X) + b$ we do **not** have $V(aX + b) = aV(X) + b$. Instead, we ha $V(aX + b) = a^2 V(X)$.

3. $$\boxed{V(a_1 X_1 + a_2 X_2) = a_1^2 V(X_1) + a_2^2 V(X_2)}$$

where X_1 and X_2 are independent random variates.

Proof : Let $Y = a_1 X_1 + a_2 X_2$ where X_1 and X_2 are independent.

$$\therefore \quad \bar{Y} = E(a_1 X_1 + a_2 X_2)$$

$$= E(a_1 X_1) + E(a_2 X_2) \qquad \text{[By Theorem 3 of § 6]}$$

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(This result is true even if X_1, X_2 are not independent.)

$$\therefore \quad V(Y) = E[(a_1 X_1 + a_2 X_2) - \{a_1 E(X_1) + a_2 E(X_2)\}]^2$$

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Example
b such that $Y = a$

Solution : We ha

$\therefore$

And

$\therefore$

Example
independent, fi

Solution : We

1. If $E($
2. Find
3. Let
that
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36, the number

The probability distribution of X is as giv

X	:	1	2	3	4	5	6
P(X = x)	:	1/36	9/36	7/36	5/36	3/36	1/36

$$E(X)=\Sigma p_i x_i=\frac{11}{36}(1)+\frac{9}{36}(2)+\frac{7}{36}(3)+\frac{5}{36}(4)+\frac{3}{36}(5)+\frac{1}{36}(6)=2{\cdot}5278$$

$$E(X^2)=\frac{11}{36}(1^2)+\frac{9}{36}(2^2)+\frac{7}{36}(3^2)+\frac{5}{36}(4^2)+\frac{3}{36}(5^2)+\frac{1}{36}(6^2)$$

$$=\frac{301}{36}=8{\cdot}3611$$

$$\therefore\quad V(X)=E(X^2)-[E(X)]^2=8{\cdot}3611-(2{\cdot}5278)^2=1{\cdot}97$$

Example 2 : A discrete random variable has the probability density function given below.

X	:	−2	−1	0	1	2	3
P(X = x)	:	0·2	k	0·1	$2k$	0·1	$2k$

Find k, the mean and variance. **(M.U. 1997, 20**

Solution : We must have $\Sigma p_i = 1$.

$$\therefore\quad 5k+0{\cdot}4=1\quad\therefore\quad 5k=0{\cdot}6\quad\therefore\quad k=\frac{0{\cdot}6}{5}=\frac{3}{25}$$

Hence, the probability distribution is

X	:	−2	−1	0	1	2	3
P(X = x)	:	2/10	3/25	1/10	6/25	1/10	6/25

Now, $\text{Mean}=E(X)=\Sigma p_i x_i=-\frac{4}{10}-\frac{3}{25}+0+\frac{6}{25}+\frac{2}{10}+\frac{18}{25}=\frac{60}{250}=\frac{6}{25}$

$$E(X^2)=\Sigma p_i x_i^2=\frac{2}{10}(4)+\frac{3}{25}(1)+0+\frac{6}{25}(1)+\frac{1}{10}(4)+\frac{6}{25}(9)=\frac{73}{250}$$

$$\therefore\quad \text{Variance}=\sigma^2=E(X^2)-[E(X)]^2=\frac{73}{250}-\frac{36}{625}=\frac{293}{625}.$$

$\therefore\quad \frac{A}{2}$

Solving the equations

$\therefore$ The p.d.f. is $f(x)$

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Solution : We have

$f_X(x)=$

$\therefore\quad$ Mean $\overline{X}$

$E(X^2)$

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Example 3 : A

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(M.U. 1997, 2001)

Example 2 : The distribution function of a r.v. X is given by $F_X(x) = 1 - (1 + x)\, e^{-x}$, $x \geq 0$. Find the mean and variance. **(M.U. 2011)**

Solution : We have

$$f_X(x) = \frac{d\,F_X(x)}{dx} = (1 + x)\,e^{-x} - e^{-x} = x\,e^{-x}, \quad x \geq 0$$

$$\therefore \quad \text{Mean } \overline{X} = \int_0^\infty x \cdot f_X(x)\,dx = \int_0^\infty x^2\,e^{-x}\,dx$$

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$$E(X^2) = \left[x^3(-e^{x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^\infty = 6$$

$$\therefore \quad V(X) = E(X^2) - [\,E(X)\,]^2 = 6 - 4 = 2.$$

Example 3 : A continuous random variable X has the p.d.f. $f(x) = kx^2\,e^{-x}$, $x \geq 0$. Find k, mean and variance. **(M.U. 2004)**

Solution : We must have $\int_0^\infty k\,x^2 e^{-x} \cdot dx = 1$

$$\therefore \quad k\left[x^2(-e^{-x}) - \int -e^{-x}\,2x\,dx\right]_0^\infty = 1$$

$$\therefore \quad k\left[-x^2 e^{-x} + 2x(-e^{-x}) - \int -2e^{-x}dx\right]_0^\infty = 1 \qquad \text{[Integrating by parts]}$$

$$k\left[-x^2\,e^{-x} - 2x\,e^{-x} - 2e^{-x}\right]_0^\infty = 1$$

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&= a_1^2\, E\{X_1 - E(X_1)\}^2 + a_2^2\, E\{X_2 - E(X_2)\}^2 + 2a_1 a_2\, E\left[\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}\right]
\end{aligned}$$

Example 3 .

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Now, there are 3 white and 7 black balls in the second urn. Probability of drawing a white ball

$$P(W) = \frac{3}{7}.$$

Hence, the required probability

$$= \frac{15}{26} \cdot \frac{1}{2} + \frac{5}{13} \cdot \frac{4}{9} + \frac{1}{26} \cdot \frac{3}{7}$$

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Example 11 : A student takes his examination in four subjects A, B, C and D. He estimates his chances of passing in A as 4 / 5, in B as 3 / 4, in C as 5 / 6 and D as 2 / 3. To qualify he must pass in A and in at least two other subjects. What is the probability that he will qualify ? **(M.U. 1998, 2000, 02, 05)**

Solution : When he qualifies, one of the following events must take place.

E_1 : He passes in A and in B, in C, in D.

E_2 : He passes in A and in B, in C but not in D.

Find the probability ...
(ii) unskilled given that she is female.

8. A box contains four tickets with numbers 111, 122, 212, 221 ... random. Let A_i $(i = 1, 2, 3)$ be the event that the i-th digit on the ticket is 1. Show that the events A_1, A_2, A_3 are pairwise independent.

9. If $P(A) = k$, $P(B) = 0·4$, $P(A \cup B) = 0·7$, find k if A, B are (i) exclusive, (ii) independent.

[Ans. : (i) 0·3, (ii) 0·5

Miscellaneous Examples

Example 1 : In throwing a fair dice, A = (The outcome is greater than 3), B = (The outcome is an even number). Find $P(A)$, $P(B)$, $P(A/B)$, $P(\bar{A} \cap B)$. **(M.U. 2004)**

Solution : We have $A = \{4, 5, 6\}$, $B = \{2, 4, 6\}$. Then $A \cap B = \{4, 6\}$

$$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{1}{6}$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example 2 : Suppose an urn contains five white balls and seven red balls. Two balls are drawn at random from the run without replacement.

(i) what is the probability that both balls are white ?

(ii) what is the probability that the second ball is red ? **(M.U. 2004)**

Solution : (i) Two white balls can be drawn as the first white and the second white.

$$P(\text{first white}) = \frac{5}{12}; \quad P(\text{second white}) = \frac{4}{11}$$

$$\therefore \quad P(\text{combined event}) = \frac{5}{12} \cdot \frac{4}{11} = \frac{5}{33}.$$

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engineer

Solution : There are in all 10 persons of whom 4 a

$\therefore \quad n =$

(i) When one Engineer of each branch is inc

This can be done in ${}^3C_1 \times {}^4C_1 \times {}^2C_1 \times$

$$\therefore \quad P(A) = \frac{{}^3C_1 \times {}^4C_1 \times {}^2C_1 \times {}^1C_1}{{}^{10}C_4} =$$

(ii) When at least one Electronics Engine

Electronics Engineer	Othe
1	
2	
3	

$\therefore$ Required probability, $P(B)$

Alternatively : We may solve this

P (At least one Electron

$= 1 - P$ (No Elec

$$= 1 - \frac{{}^7C_4}{{}^{10}C_4} = 1$$

$$= 1 - \frac{1}{6} = \frac{5}{6}.$$

on given below.

(M.U. 1997, 2001)

$\frac{3}{5}$

$\frac{3}{50}$

10-08-

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(M.U. 2011)

Solution : We have

$$f_X(x) = \frac{d\,F_X(x)}{dx} = (1+x)\,e^{-x} - e^{-x} = x\,e^{-x}, \quad x \geq 0$$

$$\therefore \quad \text{Mean } \overline{X} = \int_0^\infty x \cdot f_X(x)\,dx = \int_0^\infty x^2\,e^{-x}\,dx$$

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of 36, the number ... of ...

The probability distribution of X is as given

X	:	1	2	3	4	5	6
$P(X=x)$	:	1/36	9/36	7/36	5/36	3/36	1/36

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X	:	-2	-1	0	1	2	3
$P(X=x)$	:	$0{\cdot}2$	k	$0{\cdot}1$	$2k$	$0{\cdot}1$	$2k$

(M.U. 1997, 200

Find k, the mean and variance.

Solution : We must have $\Sigma p_i = 1$.

$$\therefore\quad 5k+0{\cdot}4=1\quad\therefore\quad 5k=0{\cdot}6\quad\therefore\quad k=\frac{0{\cdot}6}{5}=\frac{3}{25}$$

Hence, the probability distribution is

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$$\therefore\quad \text{Variance}=\sigma^2=E(X^2)-[E(X)]^2=\frac{73}{250}-\frac{36}{625}=\frac{293}{625}.$$

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Example 3 : A

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Solution : We must

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[Ans. : (i) 3 / 13, (ii) 1/

bility 1/4. Let $A = \{1,$

dent.

221, 222 and one ticket

digit on the ticket drawn

ven that he is skilled

: (i) 5 / 12, (ii) 2 / 7

drawn from the box a

hat the events A_1, A_2

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Alternatively Two balls out of 12 can be drawn in ${}^{12}C_2$ ways.

Two white balls out of 5 can be drawn in ${}^{5}C_2$ ways.

$$\therefore \quad \text{Required Probability} = \frac{{}^{5}C_2}{{}^{12}C_2} = \frac{5 \cdot 4}{12 \cdot 11} = \frac{5}{33}.$$

(ii) Second red ball can be drawn in two ways.

A = the first is white and the second is red.

B = the first is red and the second is red.

$$P(A) = \frac{5}{12} \cdot \frac{7}{11}, \quad P(B) = \frac{7}{12} \cdot \frac{6}{11}$$

$$\therefore \quad \text{Required Probability} = \frac{35}{132} + \frac{42}{132} = \frac{77}{132}.$$

Example 3 : The authorities of an university wish to form a committee of 4 engineers for inspecting the functioning of engineering colleges. 3 Electronics, 4 Mechanical, 2 Civil and 1 Computer Engineers showed interest to work as committee members. Find the probability that the committee consists of (i) one engineer of each branch, (ii) at least one Electronics Engineer.

Solution : There are in all 10 persons of whom 4 are to be selected. This can be done in ${}^{10}C_4$ ways.

$$\therefore \quad n = {}^{10}C_4.$$

(i) When one Engineer of each branch is included.

This can be done in ${}^{3}C_1 \times {}^{4}C_1 \times {}^{2}C_1 \times {}^{1}C_1$ ways.

$$\therefore \quad P(A) = \frac{{}^{3}C_1 \times {}^{4}C_1 \times {}^{2}C_1 \times {}^{1}C_1}{{}^{10}C_4} = \frac{3 \times 4 \times 2 \times 1}{10 \times 9 \times 8 \times 7} \cdot 4 \times 3 \times 2 \times 1 = \frac{4}{35}.$$

9

(3) Both black : $P(2\text{ Black}) = \dfrac{{}^{3}C_2}{{}^{13}C_2} = \dfrac{3 \cdot 2}{13 \cdot 12} = \dfrac{1}{26}$.

Now, there are 3 white and 7 black balls in the second urn. Probability of drawing a white ball

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$$= \frac{1}{9}[0 - 0 + 0 + 54] = 6$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\int_{6}^{\infty} x \cdot e^{-x/3} \cdot dx = \frac{1}{9}\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_6^{\infty}$$

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Solution : When he qualifies, one of the following events must take place.

E_1 : He passes in A and in B, in C, in D.

E_2 : He passes in A and in B, in C but not in D.

Example 3 : Find $E(X)$ if X has the p.d.f. $f(x) = \begin{cases} k\,x(2-x), & 0 \le x \le 2,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$

Solution : By definition $E(X) = \int_0^2 x\,f(x)\,dx = \int_0^2 x \cdot k\,x\,(2-x)\,dx = k\int_0^2 (2x^2 - x^3)\,dx$

$$= k\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = k\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{16k}{12} = \frac{4k}{3}.$$

Example 4 : Find $E(X)$ for the p.d.f. $f(x) = \begin{cases} k(x - x^2), & 0 \le x \le 1,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$

Solution : By definition $E(x) = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot k(x - x^2)\,dx = k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{k}{12}.$

Example 5 : The daily consumption of electric power (in million kwh) is a random variable X with probability density function

$$f(x) = \begin{cases} kx\,e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the value of k, the expectation of k and the probability that on a given day the electric consumption is more than expected value. **(M.U. 2003, 04)**

Solution : We must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$ *i.e.* $k\int_0^{\infty} x\,e^{-x/3}\,dx = 1$

$$\therefore \quad k\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^{\infty} = 1$$

$$\therefore \quad k[0 + 9] = 1 \quad \therefore\ 9k = 1 \quad \therefore\ k = 1/9$$

Alternatively Two balls out of 12 can be drawn in ${}^{12}C_2$ ways.

Two white balls out of 5 can be drawn in ${}^{5}C_2$ ways.

$$\therefore \quad \text{Required Probability} = \frac{{}^{5}C_2}{{}^{12}C_2} = \frac{5 \cdot 4}{12 \cdot 11} = \frac{5}{33}.$$

(ii) Second red ball can be drawn in two ways.

A = the first is white and the second is red.

B = the first is red and the second is red.

$$P(A) = \frac{5}{12} \cdot \frac{7}{11}, \quad P(B) = \frac{7}{12} \cdot \frac{6}{11}$$

$$\therefore \quad \text{Required Probability} = \frac{35}{132} + \frac{42}{132} = \frac{77}{132}.$$

Example 3 : The authorities of an university wish to form a committee of 4 engineers for inspecting the functioning of engineering colleges. 3 Electronics, 4 Mechanical, 2 Civil and 1 Computer Engineers showed interest to work as committee members. Find the probability that the committee consists of (i) one engineer of each branch, (ii) at least one Electronics Engineer.

Solution : There are in all 10 persons of whom 4 are to be selected. This can be done in ${}^{10}C_4$ ways.

$$\therefore \quad n = {}^{10}C_4.$$

(i) When one Engineer of each branch is included.

This can be done in ${}^{3}C_1 \times {}^{4}C_1 \times {}^{2}C_1 \times {}^{1}C_1$ ways.

$$\therefore \quad P(A) = \frac{{}^{3}C_1 \times {}^{4}C_1 \times {}^{2}C_1 \times {}^{1}C_1}{{}^{10}C} = \frac{3 \times 4 \times 2 \times 1}{10 \times 9 \times 6 \times 7} \cdot 4 \times 3 \times 2 \times 1 = \frac{4}{35}.$$

Example : If $P(A) = 0{\cdot}3$, $P(B) = 0{\cdot}5$, find $P(A \cup B)$ when (i) A, B are exclusive (ii) A, B are indepen

Solution : (i) If A, B are exclusive $P(A \cap B) = 0$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0{\cdot}3 + 0{\cdot}5 - 0 = 0{\cdot}8.$$

(ii) If A, B are independent $P(A \cap B) = P(A) \times P(B)$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$$

$$= 0{\cdot}3 + 0{\cdot}5 - 0{\cdot}3 \times 0{\cdot}5 = 0{\cdot}65.$$

(**Note** that $P(A \cup B)$ has two different values under the two conditions, showing that independe and exclusiveness are two different concepts.)

(b) Mutual Independence

If A, B, C are any three events then they are said to be **mutually independent if they are pairw** independent and if A is independent of $B \cap C$, B is independent of $C \cap A$, C is independent of $A \cap B$.

The last condition gives,

$$P[C \cap (A \cap B)] = P(C) \cdot P(A \cap B)$$

$$= P(C) \cdot P(A) \cdot P(B) \qquad [\because A, B \text{ are independent.}]$$

For the other two pairs, we get the same condition.

Thus, three events A, B, C are mutually independent if

1. $P(A \cap B) = P(A) \cdot P(B)$
2. $P(B \cap C) = P(B) \cdot P(C)$
3. $P(C \cap A) = P(C) \cdot P(A)$
4. $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$.

Generalisation : If $A_1, A_2, A_3 \ldots\ldots A_n$ are n events then they are said to be **mutually independent** if

(1) $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$; $P(A_1 \cap A_3) = P(A_1) \cdot P(A_3) \ldots\ldots$

i.e. $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ for all i, j, $i \neq j$ and

(2) $P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n) = P(A_1) \cdot P(A_2) \ldots\ldots P(A_n)$

Note .... ✍

It should be noted that the condition $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ alone **dose not** ensure pairwise independence and also the pairwise independence alone **does not** ensure $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$.

(c) Some Theorems

We now prove some theorems on conditional prob

Now, consider,

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$= 1 - \{P(A) + P(B) - P(A$

$= 1 - P(A) - P(B) + P(A$

$= [1 - P(A)] - P(B)[1 -$

$$= [1 - P(A)][1 - P(B)]$$

$$= P(\bar{A}) \cdot P(\bar{B})$$

But $P(\bar{A} / \bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \dfrac{P(\bar{A})}{P}$

Note .... ✍

Note that in words this theorem state independent".

Theorem 2 : If A, B are independer

Proof : We have $P(A \cup B) = P(A$

Since, A, B are independent $P(A \cap$

$\therefore \quad P(A \cup B) = P(A) + P(B$

$= P(A) + P($

$= 1 - P(\bar{A})$

Theorem 3 : If A, B are indepe

Proof : We have to prove that

Now, $P(A \cap \bar{B}) = P(A) - P($

$= P(A) - P($

$\therefore \quad P(A \cap \bar{B}) = P(A)[1 -$

$= P(A) \cdot P$

$\therefore \quad A, \bar{B}$ are independe

Remark .... ✍

If A, B are independent the

since $A \cap \bar{B} = A$, $P(A \cap \bar{B}) = P$

Find the probability that ...

(ii) unskilled given that she is female.

8. A box contains four tickets with numbers 111, 122, 212, 221 and ... random. Let A_i $(i = 1, 2, 3)$ be the event that the i-th digit on the ticket is 1. Show that the events A_1, A_2, A_3 are pairwise independent.

9. If $P(A) = k$, $P(B) = 0{\cdot}4$, $P(A \cup B) = 0{\cdot}7$, find k if A, B are (i) exclusive, (ii) independent.

[**Ans. : (i) 0·3, (ii) 0·5**

Miscellaneous Examples

Example 1 : In throwing a fair dice, A = (The outcome is greater than 3), B = (The outcome is an even number). Find $P(A)$, $P(B)$, $P(A/B)$, $P(\bar{A} \cap B)$. **(M.U. 2004)**

Solution : We have $A = \{4, 5, 6\}$, $B = \{2, 4, 6\}$. Then $A \cap B = \{4, 6\}$

$$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, \qquad P(B) = \frac{3}{6} = \frac{1}{2}, \qquad P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{1}{6}$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example 2 : Suppose an urn contains five white balls and seven red balls. Two balls are drawn at random from the run without replacement,

(i) what is the probability that both balls are white ?

(ii) what is the probability that the second ball is red ? **(M.U. 2004)**

Solution : (i) Two white balls can be drawn as the first white and the second white.

$$P(\text{first white}) = \frac{5}{12}\ ; \quad P(\text{second white}) = \frac{4}{11}$$

$$\therefore \quad P(\text{combined event}) = \frac{5}{12} \cdot \frac{4}{11} = \frac{5}{33}.$$

---

engineer ...

Solution : There are in all 10 persons of whom 4 a

$\therefore \quad n =$

(i) **When one Engineer of each branch is in**

This can be done in $^3C_1 \times {^4C_1} \times {^2C_1} \times$

$$\therefore \quad P(A) = \frac{^3C_1 \times {^4C_1} \times {^2C_1} \times {^1C_1}}{^{10}C_4} =$$

(ii) **When at least one Electronics Engine**

Electronics Engineer	Oth
1	
2	
3	

$\therefore$ Required probability, $P(B)$

Alternatively : We may solve thi

P (At least one Electron

$= 1 - P$ (No Elec

$$= 1 - \frac{^7C_4}{^{10}C_4} = 1$$

$$= 1 - \frac{1}{6} = \frac{5}{6}.$$

$\therefore \quad P(\overline{A}) = 1 - P(A)$

Note ....

Since A and $\overline{A}$ are exclusive, we have $A \cup \overline{A} = S$ and $A \cap \overline{A} = \Phi$.

Corollary : Probability of an event is always less than or equal to one.

i.e. $\boxed{P(A) \leq 1}$

Proof : $P(A) = 1 - P(\overline{A})$. But $P(\overline{A}) \geq 0$ by axiom 1. $\therefore \quad P(A) \leq 1$.

Remark ....

Not Both and Both Not A and B : Note that for the events A and B 'that **not both** will happen' a 'that **both will not** happen' have significantly different meanings. The first is symbolised as $\overline{A \cap B}$ and t second is symbolised as $\overline{A} \cap \overline{B}$.

Example 1 : What is the chance that a leap year selected at random will contain 53 Sundays ?

(M.U. 1996, 2003, 04, 05

Solution : A leap year has 52 complete weeks and 2 more days. These days can be Monday and Tuesday, Tuesday and Wednesday, Wednesday and Thursday, Thursday and Friday, Friday and Saturday, Saturday and Sunday, Sunday and Monday. Let the required event be denoted by A.

There are 7 outcomes and 2 are in A.

$$\therefore \quad n = 7 \text{ and } m = 2 \qquad \therefore \quad P(A) = \frac{m}{n} = \frac{2}{7}.$$

Example 2 : In the play of two dice, a person loses if the sum obtained is 2 or 4 or 12. He wins if the sum is 5 or 11. Find the ratio of his probability of losing to the probability of winning in the first throw.

Solution : When two dice are thrown simultaneously there are 36 cases as shown below,

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6).

The sum 2 is obtained in only one way - (1, 1). The sum 4 is obtained in 3 ways - (1, 3), (2, 2), (3, 1). The sum 12 is obtained in only one way - (6, 6).

$$\therefore \quad m = 1 + 3 + 1 = 5, \qquad n = 36 \qquad \therefore \quad P(\text{losing}) = \frac{m}{n} = \frac{5}{36}.$$

The sum 5 is obtained in 4 ways - (1, 4), (2, 3), (3, 2), (4, 1). The sum 11 is obtained in 2 ways - (5, 6), (6, 5).

$$\therefore \quad m = 4 + 2 = 6, \qquad n = 36 \qquad \therefore \quad P(\text{winning}) = \frac{m}{n} = \frac{6}{36}.$$

Ratio of probability of losing to probability of winning $= \dfrac{P(\text{losing})}{P(\text{winning})} = \dfrac{5/36}{6/36} = \dfrac{5}{6}$.

Example 3 : Seven persons including A and B stand
hat there are exactly two persons between A and B.

Solution : Seven persons can stand in a row in $^7C_7$ ways

If there are two persons between A and B, there are

A * * B *
* A * * B
* * A * *
* * * A *

But in these 4 ways A and B can interchange the p
5 persons can occupy the positions in $^5C_5$ ways = 5! w

$\therefore$ The number of favourable cases $m = 4 \times 2 \times$

$\therefore \quad P(A) = \frac{m}{n} = \frac{4 \times 2}{7}$

Example 4 : A fair dice is thrown thrice. Find the

Solution : A dice can be thrown 3 times in $6 \times 6 \times 6$

We can calculate the number of favourable ca

(1, 4, 5) in 3 ! = 6 ways, (1, 6, 3) in 3

(2, 5, 3) in 3 ! = 6 ways, (2, 6, 2) in

Total No. of ways of getting the sum 10 with

$\therefore \quad P(A) =$

Example 5 : A bag contains 50 tickets num
and arranged in ascending order ($x_1 < x_2 < x_3 <$

Solution : Five tickets out of 50 can be drawn i

Since x_3 is 30 and tickets are arranged in
tickets from 1 to 29 can be drawn in $^{29}C_2$ way
Thus, these two tickets can be drawn in $^{20}C_2$

$\therefore \quad m = (^{29}C_2)$

$\therefore \quad P(A) = \frac{m}{n}$

Example 6 : If n biscuits are distribute
particular child receives r biscuits where $r <$

Solution : The first biscuit can be given to a
given in m ways. Two biscuits can be given
$m \times m \times$ ..... n times $= m^n$ ways.

E_3 : He passes in A and in B, in D but not in C.

E_4 : He passes in A and in C, in D, but not in B.

Probabilities of these events are

$$P(E_1) = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{2}{3} = \frac{1}{3}$$

$$P(E_2) = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\left(1-\frac{2}{3}\right) = \frac{1}{6}$$

$$P(E_3) = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\left(1-\frac{5}{6}\right) = \frac{1}{15}$$

$$P(E_4) = \frac{4}{5}\cdot\frac{5}{6}\cdot\frac{2}{3}\cdot\left(1-\frac{3}{4}\right) = \frac{1}{9}$$

The required probability is the sum of these probabilities.

$$\begin{aligned} \therefore\quad P(\text{He qualifies}) &= P(E_1 \cup E_2 \cup E_3 \cup E_4) \\ &= P(E_1) + P(E_2) + P(E_3) + P(E_4) \\ &= \frac{1}{3} + \frac{1}{6} + \frac{1}{15} + \frac{1}{9} = \frac{61}{90}. \end{aligned}$$

Example 12 : Three news papers are published from a city. It is estimated that 20% persons read A 16% read B, 14% read C, 5% read A and C, 4% read B and C and 2% read all the three newspapers. Wha is the probability that a randomly chosen person (i) reads none of these papers, (ii) reads at least one these papers, (iii) reads only one of these papers, (iv) reads only two of these papers, (v) reads A and B bu not C ?

Solution : Problems of this type are easily solved by using Venn-diagram.

(i) P(reads at least one paper) $= 1 - \frac{65}{100} = 0.35$

(ii) P(reads none) = 0.65

(iii) P(reads only one) $= \frac{9+6+7}{}$

$$\therefore\quad P(E) = P(A \cap \overline{B}) + P($$

$(\because$ $A, B, \overline{A}, \overline{B}$ are independent ev

$$\therefore\quad P(E) = \frac{6}{10}\cdot\frac{3}{10} + \frac{4}{10}\cdot$$

Example 14 : A lot contains 1 pe andom sample so that the probability o

Solution : p = probability of a defecti

q = probability of a non-de

Probability of all non-defective ite

$$= \left(\frac{99}{100}\right)\left(\right.$$

Probability of at least one defect

We want this probability to be gr

$$\therefore\quad 1-\left(\frac{99}{100}\right)^n >$$

$$\therefore\quad -\left(\frac{99}{100}\right)^n >$$

$$\therefore\quad n\log(0.99) <$$

$$\therefore\quad n(-0.0044)$$

Example 9 : A certain test for a particular cancer is known to be 95% accurate. A person subm the test and the result is positive. Suppose that a person comes from a population of 100,000 where people suffer from that disease. What can we conclude about the probability that the person under tes that particular cancer ? **(M.U. 2008**

Solution : We have

p_1 = probability that a person has the cancer

$$= \frac{2000}{100{,}000} = \frac{2}{100} = 0{\cdot}02$$

p_2 = probability that a person does not have the cancer

$$= 1 - 0{\cdot}02 = 0{\cdot}98$$

p'_1 = probability that the test is positive when a person has the cancer

$$= \frac{95}{100} = 0{\cdot}95$$

p'_2 = probability that the test is positive when a person does not have the cancer

$$= 1 - 0{\cdot}95 = 0{\cdot}05.$$

$$\therefore \text{ Required probability } = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2}$$

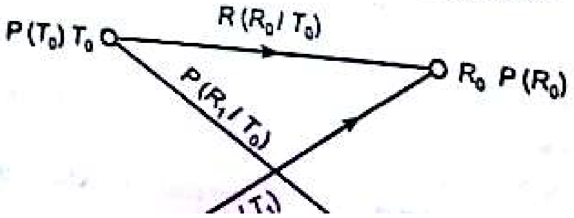
$$\therefore \text{ Required probability } = \frac{\frac{2}{100}\cdot\frac{95}{100}}{\frac{2}{100}\cdot\frac{95}{100} + \frac{98}{100}\cdot\frac{5}{100}} = \frac{190}{680} = 0{\cdot}279.$$

Example 10 : In a communication system a zero is transmitted with probability 0·4 and a one i transmitted with probability 0·6. Due to noise in the channel a zero can be received as one with probabili 0·1 and as a zero with probability 0·9. Similarly, one can be received as zero with probability 0·1 and as : one with probability 0·9.

(i) If a one is observed, what is the probability that a zero was transmitted ?

(ii) If a one is observed what is the probability that one was transmitted ? **(M.U. 2010**

Solution :



$P(R_1/T_1)$ = a 1 is rec~~e~~ived when

$P(R_0/T_1)$ = a 0 is received when

(i) P(1 is received) = P(1 is received when 1 is tr

$$P(R_1) = P(R_1/T_1)\cdot P(T_1) + P(R_1/$$

$$= 0{\cdot}9 \times 0{\cdot}6 + 0{\cdot}1 \times 0{\cdot}4 = 0{\cdot}5$$

(ii) P(0 is received) = P(0 is received when 0 is tr

$$P(R_0) = P(R_0/T_0)\cdot P(T_0) + P(R$$

$$= 0{\cdot}9 \times 0{\cdot}4 + 0{\cdot}1 \times 0{\cdot}6 = 0{\cdot}4$$

(iii) P(0 was transmitted given that 1 was receive

$$P(T_0/R_1) = \frac{P(R_1/T_0)\cdot P(T_0)}{P(R_1)} = \frac{0{\cdot}1\times}{0{\cdot}}$$

(iv) P(1 was transmitted given that 0 was recei

$$P(T_1/R_1) = \frac{P(R_1/T_1)\cdot P(T_1)}{P(R_1)} = \frac{0{\cdot}9}{0}$$

Alternatively : $P(T_1/R_1) = 1 - P(T_0/R_1)$

Example 11 : A binary communication tra by 0 or 1. Due to noise, sometimes a transmitte

If the probability that a transmitted 0 is corr 1 is correctly received as 1 is 0·8 and if the pr a 1 is received, (ii) a 0 is received, (iii) a 1 was tr given that a 0 was received, (v) the error has

Solution : We are given that

$P(T_0)$ = a 0 is transmitted

$P(T_1)$ = a 1 is transmitted

$= 1 - p_1 = 0{\cdot}55$

$P(R_0/T_0)$ = a 0 is received w

$P(R_1/T_0)$ = a 1 received whe

$= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1/T_1)$ = a 1 is received

$P(R_0/T_1)$ = a 0 is received

$= 1 - 0{\cdot}8 = 0{\cdot}2.$

p_1' = answer is correct (by

p_2' = answering is correct (by guessing) = $\frac{1}{4}$

Required Probability $= \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2'} = \frac{(1/10)\cdot(1/4)}{(9/10)\cdot 1 + (1/10)\cdot(1/4)}$

$= \frac{1/40}{37/40} = \frac{1}{37}$

Example 6 : A bag contains five balls, the colours of which are not known. Two balls were draw the bag and they were found to be white. What is the probability that all balls are white ? **(M.U.**

Solution : Since two balls drawn are white, the bag may contain 2 white or 3 white or 4 white or 5 balls.

Let these events be denoted by A_1, A_2, A_3, A_4 respectively. We can assume that the probabilit these events are equal.

Let $p_1 = P(A_1)$, $p_2 = P(A_2)$, $p_3 = P(A_3)$, $p_4 = P(A_4)$.

$\therefore \quad p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$

Now, two balls out of 5 can be drawn in ${}^5C_2$ ways.

$\therefore \quad p_1'$ = P (drawing two white balls when two balls are white) $= \frac{{}^2C_2}{{}^5C_2} = \frac{2\cdot 1}{5\cdot 4} = \frac{2}{20}$.

p_2' = P (drawing two white balls when 3 balls are white) $= \frac{{}^3C_2}{{}^5C_2} = \frac{3\cdot 2}{5\cdot 4} = \frac{6}{20}$.

p_3' = P (drawing 2 white balls when 4 balls are white) $= \frac{{}^4C_2}{{}^5C_2} = \frac{4\times 3}{5\times 4} = \frac{12}{20}$.

p_4' = P (drawing 2 white balls when 5 balls are white) $= \frac{{}^5C_2}{{}^5C_2} = \frac{5\cdot 4}{5\cdot 4} = \frac{20}{20}$

$\therefore$ By Baye's theorem Required Probability

$$= \frac{p_4 p_4'}{p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'}$$

$$= \frac{(1/4)\cdot(20/20)}{(1/4)\cdot(2/20) + (1/4)\cdot(6/20) + (1/4)\cdot(12/20) + (1/4)\cdot(20/20)}$$

$$= \frac{20}{2+6+12+20} = \frac{20}{40} = \frac{1}{2}.$$

Example 7 : The contents of three urns are as follows :

2 white balls, 3 black balls and 5 red balls,
3 white balls, 4 black balls and 3 red balls,
5 white balls, 3 black balls and 2 red balls,

One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urn 1 or 2 or 3 respectively ?

C_1 = drawing urn 1, $\quad C_2$ =

$P(C_1) = P(C_2) = P(C_3) =$

$\therefore \quad P\left(\frac{C_1}{B}\right) = \frac{2/9}{(2/9) + (1/5) + (2}$

$P\left(\frac{C_2}{B}\right) = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2' +}$

$= \frac{(1}{(1/3)\cdot(2/9) + (1}$

$P\left(\frac{C_3}{B}\right) = \frac{p_3 p_2'}{p_1 p_1' + p_2 p_2' +}$

$= \frac{(1}{(1/3)\cdot(2/9) + (}$

Example 8 : A lot of IC chips is delivery but the tester is not complete

P (Tester says the chip is good defective / The chip is actually defect

If a tested chip is declared defec

Solution : Let $\quad p_1$ = Chip is defecti

p_1' = Tester says th

p_2 = Chip is good =

p_2' = Tester says th

By Bayes' Theorem,

P (Chip is defective / Tester

$$\therefore \quad P(R_0) = P(R_0/T_0)\cdot P(T_0) + P(R_0/T_1)\cdot P(T_1)$$
$$= 0{\cdot}9 \times 0{\cdot}45 + 0{\cdot}2 \times 0{\cdot}55 = 0{\cdot}515$$

Now, by Bayes' Theorem

(iii) P(1 was transmitted given that 1 was received) *i.e.*

$$P(T_1/R_1) = \frac{P(R_1/T_1)\,P(T_1)}{P(R_1)} = \frac{0{\cdot}8 \times 0{\cdot}55}{0{\cdot}485} = 0{\cdot}907$$

(iv) P(0 was transmitted given that 0 was received) *i.e.*

$$P(T_0/R_0) = \frac{P(R_0/T_0)\cdot P(T_0)}{P(R_0)} = \frac{0{\cdot}9 \times 0{\cdot}45}{0{\cdot}515} = 0{\cdot}786$$

(v) P(Error) = P(0 was received when 1 is transmitted given that 1 was transmitted)
+ P(1 was received when 0 was transmitted given that 0 was transmitted)

$$P(\text{Error}) = P(R_0/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$
$$= 0{\cdot}2 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}155.$$

Example 12 : In a factory, four machines A_1, A_2, A_3 and A_4 produce 10%, 20%, 30% and 40% o items respectively. The percentage of defective items produced by them is 5%, 4%, 3% and 2% respectiv An item is selected at random is found to be defective. What is the probability that it was produced by t machine A_2?

(M.U. 20

Solution : Let $p_1 = P(A_1)$, $p_2 = P(A_2)$, etc., then we have

$$p_1 = 0{\cdot}1, \quad p_2 = 0{\cdot}2, \quad p_3 = 0{\cdot}3, \quad p_4 = 0{\cdot}4$$

Let $p'_1 = P(B/A_1)$, $p'_2 = P(B/A_2)$, etc., then we have

$$p'_1 = 0{\cdot}05, \quad p'_2 = 0{\cdot}04, \quad p'_3 = 0{\cdot}03, \quad p'_4 = 0{\cdot}02$$

Now, required probability
$$= \frac{p_2\,p'_2}{p_1 p'_1 + p_2 p'_2 + p_3 p'_3 + p_4 p'_4}$$
$$= \frac{0{\cdot}2 \times 0{\cdot}04}{0{\cdot}1 \times 0{\cdot}05 \times 0{\cdot}2 \times 0{\cdot}04 + 0{\cdot}3 \times 0{\cdot}03 + 0{\cdot}4 \times 0{\cdot}02}$$
$$= \frac{0{\cdot}008}{0{\cdot}03} = 0{\cdot}27.$$

---

p_1' = Probability of getting 2 heads i

$$= {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3\cdot\frac{1}{9}\cdot\frac{2}{3} = \frac{2}{9}$$

p_2' = Probability of getting 2 heads

$$= {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3\cdot\frac{4}{9}\cdot\frac{1}{3} =$$

p_3' = Probability of getting 2 hea

$$= {}^3C_2\left(\frac{3}{4}\right)^2\cdot\left(\frac{1}{4}\right) = 3\cdot\frac{9}{16}\cdot$$

$\therefore$ Required Probability $= \dfrac{\quad}{p_1 p'_1 +}$

$$= \frac{(1/3)(}{(1/3)\cdot(2/9) + (1/3)\cdot(}$$

$$= \frac{(2/9)}{(2/9) + (4/9) + (27/}$$

(b) We do not know which coin was tos

If the coin was A, the probability tha

= (Prob. of choosing A) ×

$$= \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}.$$

If the coin was B, probability of ge

= (Prob. of choosing B)

$$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}.$$

If the coin was C, probability of

= (Prob. of choosing

p_1' = answer is correct ...

p_2' = answering is correct (by guessing) = $\frac{1}{4}$

$$\text{Required Probability} = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2'} = \frac{(1/10)\cdot(1/4)}{(9/10)\cdot 1 + (1/10)\cdot(1/4)}$$

$$= \frac{1/40}{37/40} = \frac{1}{37}$$

Example 6 : A bag contains five balls, the colours of which are not known. Two balls were drawn the bag and they were found to be white. What is the probability that all balls are white ? **(M.U.**

Solution : Since two balls drawn are white, the bag may contain 2 white or 3 white or 4 white or 5 balls.

Let these events be denoted by A_1, A_2, A_3, A_4 respectively. We can assume that the probabili these events are equal.

Let $p_1 = P(A_1)$, $p_2 = P(A_2)$, $p_3 = P(A_3)$, $p_4 = P(A_4)$

$$\therefore \quad p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$$

Now, two balls out of 5 can be drawn in ${}^5C_2$ ways.

$\therefore$ p_1' = P (drawing two white balls when two balls are white) $= \frac{{}^2C_2}{{}^5C_2} = \frac{2\cdot 1}{5\cdot 4} = \frac{2}{20}$.

p_2' = P (drawing two white balls when 3 balls are white) $= \frac{{}^3C_2}{{}^5C_2} = \frac{3\cdot 2}{5\cdot 4} = \frac{6}{20}$.

p_3' = P (drawing 2 white balls when 4 balls are white) $= \frac{{}^4C_2}{{}^5C_2} = \frac{4\times 3}{5\times 4} = \frac{12}{20}$.

p_4' = P (drawing 2 white balls when 5 balls are white) $= \frac{{}^5C_2}{{}^5C_2} = \frac{5\cdot 4}{5\cdot 4} = \frac{20}{20}$

$\therefore$ By Baye's theorem Required Probability

$$= \frac{p_4 p_4'}{p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'}$$

$$= \frac{(1/4)\cdot(20/20)}{(1/4)\cdot(2/20) + (1/4)\cdot(6/20) + (1/4)\cdot(12/20) + (1/4)\cdot(20/20)}$$

$$= \frac{20}{2+6+12+20} = \frac{20}{40} = \frac{1}{2}.$$

Example 7 : The contents of three urns are as follows :

2 white balls, 3 black balls and 5 red balls,
3 white balls, 4 black balls and 3 red balls,
5 white balls, 3 black balls and 2 red balls,

One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urn 1 or 2 or 3 respectively ?

C_1 = Choosing urn 1, C_2 =

$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$

$P\left(\frac{B}{C_1}\right) = p_1' = \frac{{}^2C_1 \cdot {}^5C_1}{{}^{10}C_2}$

$P\left(\frac{B}{C_2}\right) = p_2' = \frac{{}^3C_1 \cdot {}^3C_1}{{}^{10}C_2}$

$P\left(\frac{B}{C_3}\right) = p_3' = \frac{{}^5C_1 \cdot {}^2C_1}{{}^{10}C_2}$

$\therefore$ $P\left(\frac{C_1}{B}\right) = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2'}$

$= \frac{}{(1/3)\cdot(2/9)}$

$\therefore$ $P\left(\frac{C_1}{B}\right) = \frac{2/9}{(2/9) + (1/5) + (2}$

$P\left(\frac{C_2}{B}\right) = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2' +}$

$= \frac{}{(1/3)\cdot(2/9) + (}$

$P\left(\frac{C_3}{B}\right) = \frac{p_3 p_3'}{p_1 p_1' + p_2 p_2'}$

$= \frac{}{(1/3)\cdot(2/9) +}$

Example 8 : A lot of IC chips i delivery but the tester is not complete

P (Tester says the chip is good defective / The chip is actually defec

If a tested chip is declared defe

Solution : Let p_1 = Chip is defect
p'_1 = Tester says t
p_2 = Chip is good
p'_2 = Tester says

By Bayes' Theorem.

P (Chip is defective / Tester

Example 9 : A certain test for a particular cancer is known to be 95% accurate. A person subm the test and the result is positive. Suppose that a person comes from a population of 100,000 where ; people suffer from that disease. What can we conclude about the probability that the person under tes that particular cancer ?

(M.U. 200

Solution : We have

p_1 = probability that a person has the cancer

$$= \frac{2000}{100{,}000} = \frac{2}{100} = 0{\cdot}02$$

p_2 = probability that a person does not have the cancer

$$= 1 - 0{\cdot}02 = 0{\cdot}98$$

p'_1 = probability that the test is positive when a person has the cancer

$$= \frac{95}{100} = 0{\cdot}95$$

p'_2 = probability that the test is positive when a person does not have the cancer

$$= 1 - 0{\cdot}95 = 0{\cdot}05.$$

$$\therefore \text{ Required probability } = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2}$$

$$\therefore \text{ Required probability } = \frac{\frac{2}{100}\cdot\frac{95}{100}}{\frac{2}{100}\cdot\frac{95}{100} + \frac{98}{100}\cdot\frac{5}{100}} = \frac{190}{680} = 0{\cdot}279.$$

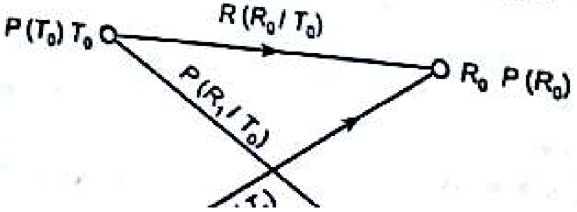
Example 10 : In a communication system a zero is transmitted with probability 0·4 and a one transmitted with probability 0·6. Due to noise in the channel a zero can be received as one with probabil 0·1 and as a zero with probability 0·9. Similarly, one can be received as zero with probability 0·1 and as one with probability 0·9.

(i) If a one is observed, what is the probability that a zero was transmitted ?

(ii) If a one is observed what is the probability that one was transmitted ?

(M.U. 2010

Solution :



$P(R_1 / T_1)$ = a 1 is received wh

$P(R_0 / T_1)$ = a 0 is received w

i) **P (1 is received)** = P (1 is received when 1 is

$$P(R_1) = P(R_1/T_1)\cdot P(T_1) + P(R$$

$$= 0{\cdot}9 \times 0{\cdot}6 + 0{\cdot}1 \times 0{\cdot}4 =$$

ii) **P (0 is received)** = P (0 is received when 0 is

$$P(R_0) = P(R_0/T_0)\cdot P(T_0) + P$$

$$= 0{\cdot}9 \times 0{\cdot}4 + 0{\cdot}1 \times 0{\cdot}6 =$$

(iii) P (0 was transmitted given that 1 was rece

$$P(T_0/R_1) = \frac{P(R_1/T_0)\cdot P(T_0)}{P(R_1)} = \frac{0\text{-}}{}$$

(iv) P (1 was transmitted given that 0 was rec

$$P(T_1/R_1) = \frac{P(R_1/T_1)\cdot P(T_1)}{P(R_1)} = \frac{0\text{-}}{}$$

Alternatively : $P(T_1 / R_1) = 1 - P(T_0 / R$

Example 11 : A binary communication tr by 0 or 1. Due to noise, sometimes a transmi

If the probability that a transmitted 0 is co 1 is correctly received as 1 is 0·8 and if the p a 1 is received, (ii) a 0 is received, (iii) a 1 was given that a 0 was received, (v) the error ha

Solution : We are given that

$P(T_0)$ = a 0 is transmitted

$P(T_1)$ = a 1 is transmitted

$= 1 - p_1 = 0{\cdot}55$

$P(R_0 / T_0)$ = a 0 is received

$P(R_1 / T_0)$ = a 1 received wh

$= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1 / T_1)$ = a 1 is received

$P(R_0 / T_1)$ = a 0 is received

E_3 : He passes in A and in B, in D but not in C.

E_4 : He passes in A and in C, in D, but not in B.

Probabilities of these events are

$$P(E_1)=\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{2}{3}=\frac{1}{3}$$

$$P(E_2)=\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\left(1-\frac{2}{3}\right)=\frac{1}{6}$$

$$P(E_3)=\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\left(1-\frac{5}{6}\right)=\frac{1}{15}$$

$$P(E_4)=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{2}{3}\cdot\left(1-\frac{3}{4}\right)=\frac{1}{9}$$

The required probability is the sum of these probabilities.

$$\begin{aligned}\therefore\quad P(\text{He qualifies}) &= P(E_1\cup E_2\cup E_3\cup E_4)\\ &= P(E_1)+P(E_2)+P(E_3)+P(E_4)\\ &= \frac{1}{3}+\frac{1}{6}+\frac{1}{15}+\frac{1}{9}=\frac{61}{90}.\end{aligned}$$

Example 12 : Three news papers are published from a city. It is estimated that 20% persons read
16% read B, 14% read C, 5% read A and C, 4% read B and C and 2% read all the three newspapers. Wh
is the probability that a randomly chosen person (i) reads none of these papers, (ii) reads at least one
these papers, (iii) reads only one of these papers, (iv) reads only two of these papers, (v) reads A and B b
not C ?

Solution : Problems of this type are easily solved by using Venn-diagram.

(i) P(reads at least one paper) $=1-\frac{65}{100}=0.35$

(ii) P(reads none) = 0.65

(iii) P(reads only one) $=\frac{9+6+7}{}$

$\therefore\quad P(E)=P(A\cap\bar{B})+P($

($\because$ $A, B, \bar{A}, \bar{B}$ are independent ev

$\therefore\quad P(E)=\frac{6}{10}\cdot\frac{3}{10}+\frac{4}{10}\cdot$

Example 14 : A lot contains 1 pe
andom sample so that the probability o

Solution : p = probability of a defecti

q = probability of a non-de

Probability of all non-defective ite

$=\left(\frac{99}{100}\right)\Big($

Probability of at least one defect

We want this probability to be gr

$\therefore\quad 1-\left(\frac{99}{100}\right)^n>$

$\therefore\quad -\left(\frac{99}{100}\right)^n>$

$\therefore\quad n\log(0.99)<$

$\therefore\quad n(-0.0044)$

s the cancer

es not have the cancer

g

d with probability 0·4 and a one is
be received as one with probability
s zero with probability 0·1 and as a

nsmitted ?

nitted ? **(M.U. 2010)**

$P(R_0)$

$P(R_1)$

0·9

0·1

(ii) P (0 is received) = P (0 is received when 0 is transmitted) + P (0 is received when 1 is

$$P(R_0) = P(R_0/T_0)\cdot P(T_0) + P(R_0/T_1)\cdot P(T_1)$$

$$= 0{\cdot}9 \times 0{\cdot}4 + 0{\cdot}1 \times 0{\cdot}6 = 0{\cdot}42$$

(iii) P (0 was transmitted given that 1 was received)

$$P(T_0/R_1) = \frac{P(R_1/T_0)\cdot P(T_0)}{P(R_1)} = \frac{0{\cdot}1 \times 0{\cdot}4}{0{\cdot}58} = \frac{0{\cdot}04}{0{\cdot}58} = 0{\cdot}07$$

(iv) P (1 was transmitted given that 0 was received)

$$P(T_1/R_1) = \frac{P(R_1/T_1)\cdot P(T_1)}{P(R_1)} = \frac{0{\cdot}9 \times 0{\cdot}6}{0{\cdot}58} = \frac{0{\cdot}54}{0{\cdot}58} = 0{\cdot}93.$$

Alternatively : $P(T_1/R_1) = 1 - P(T_0/R_1) = 1 - 0{\cdot}07 = 0{\cdot}93$.

Example 11 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred. **(M.U. 2009)**

Solution : We are given that

$P(T_0)$ = a 0 is transmitted = 0·45

$P(T_1)$ = a 1 is transmitted = $1 - P(T_0)$

$= 1 - p_1 = 0{\cdot}55$

$P(R_0/T_0)$ = a 0 is received when a 0 was transmitted = 0·9

$P(R_1/T_0)$ = a 1 received when a 0 was transmitted

$= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1/T_1)$ = a 1 is received when a 1 was transmitted = 0·8

$P(R_0/T_1)$ = a 0 is received when a 1 was transmitted

$= 1 - 0{\cdot}8 = 0{\cdot}2.$

Now, we calculate the required probabilities as follows :

(i) P (1 is received) = P (1 is received when 1 is transmitted)
+ P (1 is received when 0 is transmitted)

$$\therefore \quad P(R_1) = P(R_1/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$

$$= 0{\cdot}8 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}485$$

(ii) P (0 is received) = P (0 is received when 0 is transmitted)
+ P (0 is received when 1 is transmitted)

(i) P(1 is received) = P(1 is received when 1 is transmitted) +

$$P(R_1) = P(R_1 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0)$$
$$= 0{\cdot}9 \times 0{\cdot}6 + 0{\cdot}1 \times 0{\cdot}4 = 0{\cdot}58$$

(ii) P(0 is received) = P(0 is received when 0 is transmitted) + P(0 is received when 1 is transmitted)

$$P(R_0) = P(R_0 / T_0) \cdot P(T_0) + P(R_0 / T_1) \cdot P(T_1)$$
$$= 0{\cdot}9 \times 0{\cdot}4 + 0{\cdot}1 \times 0{\cdot}6 = 0{\cdot}42$$

(iii) P(0 was transmitted given that 1 was received)

$$P(T_0/R_1) = \frac{P(R_1/T_0) \cdot P(T_0)}{P(R_1)} = \frac{0.1 \times 0.4}{0.58} = \frac{0.04}{0.58} = 0.07$$

(iv) P(1 was transmitted given that 0 was received)

$$P(T_1/R_1) = \frac{P(R_1/T_1) \cdot P(T_1)}{P(R_1)} = \frac{0.9 \times 0.6}{0.58} = \frac{0.54}{0.58} = 0{\cdot}93.$$

Alternatively : $P(T_1 / R_1) = 1 - P(T_0 / R_1) = 1 - 0{\cdot}07 = 0{\cdot}93.$

Example 11 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred. **(M.U. 2009)**

Solution : We are given that

$P(T_0)$ = a 0 is transmitted = 0·45

$P(T_1)$ = a 1 is transmitted = $1 - P(T_0)$

$= 1 - p_1 = 0{\cdot}55$

$P(R_0 / T_0)$ = a 0 is received when a 0 was transmitted = 0·9

$P(R_1 / T_0)$ = a 1 received when a 0 was transmitted

$= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1 / T_1)$ = a 1 is received when a 1 was transmitted = 0·8

$P(R_0 / T_1)$ = a 0 is received when a 1 was transmitted

$= 1 - 0{\cdot}8 = 0{\cdot}2.$

Now, we calculate the required probabilities as follows :

(i) P(1 is received) = P(1 is received when 1 is transmitted)
+ P(1 is received when 0 is transmitted)

$$\therefore \quad P(R_1) = P(R_1 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0)$$
$$= 0{\cdot}8 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}485$$

(ii) P(0 is received) = P(0 is received when 0 is transmitted)
+ P(0 is received when 1 is transmitted)

ncer

son has the cancer

on does not have the cancer

= 0·279

mitted with probability 0·4 and a one is
o can be received as one with probability
ved as zero with probability 0·1 and as a

as transmitted ?

transmitted ? **(M.U. 2010)**

O R_0 $P(R_0)$

O R_1 $P(R_1)$

ted = 0·9
ted = 0·1

(i) P (1 is received) = P (1 is received when 1 is transmitted) +

$$P(R_1) = P(R_1/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$
$$= 0{\cdot}9 \times 0{\cdot}6 + 0{\cdot}1 \times 0{\cdot}4 = 0{\cdot}58$$

(ii) P (0 is received) = P (0 is received when 0 is transmitted) + P (0 is received when 1 is transmitted)

$$P(R_0) = P(R_0/T_0)\cdot P(T_0) + P(R_0/T_1)\cdot P(T_1)$$
$$= 0{\cdot}9 \times 0{\cdot}4 + 0{\cdot}1 \times 0{\cdot}6 = 0{\cdot}42$$

(iii) P (0 was transmitted given that 1 was received)

$$P(T_0/R_1) = \frac{P(R_1/T_0)\cdot P(T_0)}{P(R_1)} = \frac{0.1 \times 0.4}{0.58} = \frac{0.04}{0.58} = 0.07$$

(iv) P (1 was transmitted given that 0 was received)

$$P(T_1/R_1) = \frac{P(R_1/T_1)\cdot P(T_1)}{P(R_1)} = \frac{0.9 \times 0.6}{0.58} = \frac{0.54}{0.58} = 0{\cdot}93.$$

Alternatively : $P(T_1/R_1) = 1 - P(T_0/R_1) = 1 - 0{\cdot}07 = 0{\cdot}93$.

Example 11 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred. **(M.U. 2009)**

Solution : We are given that

$P(T_0)$ = a 0 is transmitted = 0·45
$P(T_1)$ = a 1 is transmitted = $1 - P(T_0)$
$= 1 - p_1 = 0{\cdot}55$
$P(R_0/T_0)$ = a 0 is received when a 0 was transmitted = 0·9
$P(R_1/T_0)$ = a 1 received when a 0 was transmitted
$= 1 - 0{\cdot}9 = 0{\cdot}1$
$P(R_1/T_1)$ = a 1 is received when a 1 was transmitted = 0·8
$P(R_0/T_1)$ = a 0 is received when a 1 was transmitted
$= 1 - 0{\cdot}8 = 0{\cdot}2.$

Now, we calculate the required probabilities as follows :

(i) P (1 is received) = P (1 is received when 1 is transmitted) + P (1 is received when 0 is transmitted)

$$\therefore \quad P(R_1) = P(R_1/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$
$$= 0{\cdot}8 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}485$$

(ii) P (0 is received) = P (0 is received when 0 is transmitted) + P (0 is received when 1 is transmitted)

$$\therefore \quad P(R_0) = P(R_0/T_0)\cdot P(T_0) + P(R_0/T_1)\cdot P(T_1)$$
$$= 0{\cdot}9 \times 0{\cdot}45 + 0{\cdot}2 \times 0{\cdot}55 = 0{\cdot}515$$

Now, by Bayes' Theorem

(iii) P (1 was transmitted given that 1 was received) *i.e.*

$$P(T_1/R_1) = \frac{P(R_1/T_1)\,P(T_1)}{P(R_1)} = \frac{0{\cdot}8 \times 0{\cdot}55}{0{\cdot}485} = 0{\cdot}907$$

(iv) P (0 was transmitted given that 0 was received) *i.e.*

$$P(T_0/R_0) = \frac{P(R_0/T_0)\cdot P(T_0)}{P(R_0)} = \frac{0{\cdot}9 \times 0{\cdot}45}{0{\cdot}515} = 0{\cdot}786$$

(v) P (Error) = P (0 was received when 1 is transmitted given that 1 was transmitted)
+ P (1 was received when 0 was transmitted given that 0 was transmitted)

$$P(\text{Error}) = P(R_0/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$
$$= 0{\cdot}2 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}155.$$

Example 12 : In a factory, four machines A_1, A_2, A_3 and A_4 produce 10%, 20%, 30% and 40% d items respectively. The percentage of defective items produced by them is 5%, 4%, 3% and 2% respectiv An item is selected at random is found to be defective. What is the probability that it was produced by t machine A_2?

(M.U. 20

Solution : Let $p_1 = P(A_1)$, $p_2 = P(A_2)$, etc., then we have

$$p_1 = 0{\cdot}1, \quad p_2 = 0{\cdot}2, \quad p_3 = 0{\cdot}3, \quad p_4 = 0{\cdot}4$$

Let $p'_1 = P(B/A_1)$, $p'_2 = P(B/A_2)$, etc., then we have

$$p'_1 = 0{\cdot}05, \quad p'_2 = 0{\cdot}04, \quad p'_3 = 0{\cdot}03, \quad p'_4 = 0{\cdot}02$$

$$\text{Now, required probability} = \frac{p_2 p'_2}{p_1 p'_1 + p_2 p'_2 + p_3 p'_3 + p_4 p'_4}$$
$$= \frac{0{\cdot}2 \times 0{\cdot}04}{0{\cdot}1 \times 0{\cdot}05 \times 0{\cdot}2 \times 0{\cdot}04 + 0{\cdot}3 \times 0{\cdot}03 + 0{\cdot}4 \times 0{\cdot}02}$$
$$= \frac{0{\cdot}008}{0{\cdot}03} = 0{\cdot}27.$$

---

p'_1 = Probability of getting 2 heads in th

$$= {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3\cdot\frac{1}{9}\cdot\frac{2}{3} = \frac{2}{9}.$$

p'_2 = Probability of getting 2 heads ir

$$= {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3\cdot\frac{4}{9}\cdot\frac{1}{3} = \frac{4}{9}.$$

p'_3 = Probability of getting 2 heads

$$= {}^3C_2\left(\frac{3}{4}\right)^2\cdot\left(\frac{1}{4}\right) = 3\cdot\frac{9}{16}\cdot\frac{1}{4} =$$

$\therefore$ Required Probability = $\frac{p}{p_1 p'_1 + p}$

$$= \frac{(1/3)(2}{(1/3)\cdot(2/9) + (1/3)\cdot(4}$$

$$= \frac{(2/9)}{(2/9) + (4/9) + (27/64}$$

(b) We do not know which coin was toss

If the coin was A, the probability that

= (Prob. of choosing A) ×

$$= \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}.$$

If the coin was B, probability of ge

= (Prob. of choosing B)

$$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}.$$

If the coin was C, probability of

= (Prob. of choosing

Example 3 : Find $E(X)$ if X has the p.d.f. $f(x) = \begin{cases} k\,x(2-x), & 0 \le x \le 2,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$

Solution : By definition $E(X) = \int_0^2 x\, f(x)\, dx = \int_0^2 x \cdot k\, x\,(2-x)\, dx = k\int_0^2 (2x^2 - x^3)\, dx$

$$= k\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = k\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{16k}{12} = \frac{4k}{3}.$$

Example 4 : Find $E(X)$ for the p.d.f. $f(x) = \begin{cases} k(x - x^2), & 0 \le x \le 1,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$

Solution : By definition $E(x) = \int_0^1 x\, f(x)\, dx = \int_0^1 x \cdot k(x - x^2)\, dx = k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{k}{12}.$

Example 5 : The daily consumption of electric power (in million kwh) is a random variable X with probability density function

$$f(x) = \begin{cases} kx\,e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the value of k, the expectation of k and the probability that on a given day the electric consumption is more than expected value. **(M.U. 2003, 04)**

Solution : We must have $\int_{-\infty}^{\infty} f(x)\, dx = 1$ *i.e.* $k\int_0^{\infty} x\,e^{-x/3}\, dx = 1$

$$\therefore \quad k\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^{\infty} = 1$$

$$\therefore \quad k[0 + 9] = 1 \quad \therefore\ 9k = 1 \quad \therefore\ k = 1/9$$

Example : If $P(A)=0{\cdot}3$, $P(B)=0{\cdot}5$, find $P(A\cup B)$ when (i) A, B are exclusive (ii) A, B are indepen

Solution : (i) If A, B are exclusive $P(A\cap B)=0$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\therefore\quad P(A\cup B)=P(A)+P(B)-0=0{\cdot}8.$$

$$=0{\cdot}3+0{\cdot}5-0=0{\cdot}8.$$

(ii) If A, B are independent $P(A\cap B)=P(A)\times P(B)$

$$\therefore\quad P(A\cup B)=P(A)+P(B)-P(A)\times P(B)$$

$$=0{\cdot}3+0{\cdot}5-0{\cdot}3\times 0{\cdot}5=0{\cdot}65.$$

(**Note** that $P(A\cup B)$ has two different values under the two conditions, showing that independe and exclusiveness are two different concepts.)

(b) Mutual Independence

If A, B, C are any three events then they are said to be **mutually independent** if they are pairwi independent and if A is independent of $B\cap C$, B is independent of $C\cap A$, C is independent of $A\cap B$.

The last condition gives,

$$P[C\cap(A\cap B)] = P(C)\cdot P(A\cap B)$$

$$= P(C)\cdot P(A)\cdot P(B) \quad [\because A, B \text{ are independent.}]$$

For the other two pairs, we get the same condition.

Thus, three events A, B, C are mutually independent if

1. $P(A\cap B)=P(A)\cdot P(B)$
2. $P(B\cap C)=P(B)\cdot P(C)$
3. $P(C\cap A)=P(C)\cdot P(A)$
4. $P(A\cap B\cap C)=P(A)\cdot P(B)\cdot P(C)$.

Generalisation : If $A_1, A_2, A_3 \ldots\ldots A_n$ are n events then they are said to be **mutually independent** if

(1) $P(A_1\cap A_2)=P(A_1)\cdot P(A_2)$; $P(A_1\cap A_3)=P(A_1)\cdot P(A_3)$ .....

i.e. $P(A_i\cap A_j)=P(A_i)\cdot P(A_j)$ for all i, j, $i\neq j$ and

(2) $P(A_1\cap A_2\cap\ldots\ldots\cap A_n)=P(A_1)\cdot P(A_2)\ldots\ldots P(A_n)$

Note ....

It should be noted that the condition $P(A\cap B\cap C)=P(A)\cdot P(B)\cdot P(C)$ alone **does not** ensure pairwise independence and also the pairwise independence alone **does not** ensure $P(A\cap B\cap C)=P(A)\cdot P(B)\cdot P(C)$.

(c) Some Theorems

We now prove some theorems on conditional probability

Theorem 1 : If

Now, consider,

$$P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})$$

$$=1-P(A\cup B)$$

$$=1-\{P(A)+P(B)-P(A$$

$$=1-P(A)-P(B)+P(A$$

$$=[1-P(A)]-P(B)[1-$$

$$=[1-P(A)][1-P(B)]$$

$$=P(\bar{A})\cdot P(\bar{B})$$

But $\quad P(\bar{A}/\bar{B})=\dfrac{P(\bar{A}\cap\bar{B})}{P(\bar{B})}=\dfrac{P(\bar{A})}{P}$

Note ....

Note that in words this theorem state independent".

Theorem 2 : If A, B are independen

Proof : We have $P(A\cup B)=P(A$

Since, A, B are independent $P(A\cap$

$$\therefore\ P(A\cup B) = P(A)+P(B$$

$$= P(A)+P($$

$$= 1-P(\bar{A})$$

Theorem 3 : If A, B are indepe

Proof : We have to prove that

Now, $\quad P(A\cap\bar{B})=P(A)-P(A$

$$=P(A)-P($$

$$\therefore\quad P(A\cap\bar{B})=P(A)[1-$$

$$=P(A)\cdot P$$

$\therefore\quad A, \bar{B}$ are independe

Remark ....

If A, B are independent the since $A\cap\bar{B}=A$, $P(A\cap\bar{B})=P$